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THE STRONG RUNNING COUPLING FROM AN APPROXIMATE GLUON DYSON-SCHWINGER EQUATION†

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Using Mandelstam's approximation to the gluon Dyson-Schwinger equation we calculate the gluon self-energy in a renormalisation group invariant fashion. We obtain a non-perturbative β function. The scaling behaviour near the ultraviolet stable fixed point is in good agreement with perturbative QCD. No further fixed point for positive values of the coupling is found: α_S increases without bound in the infrared.

The infrared behaviour of the running coupling in strong interactions, α_S, is of great interest, since it may provide an understanding of confinement. Its study is an intrinsically non-perturbative problem. One suitable framework to address this problem is provided by the Dyson-Schwinger equations of QCD. Studies of this infinite tower of equations rely on specific truncation schemes. Here we will focus on an approximation to the gluon Dyson-Schwinger equation originally proposed by Mandelstam.[1] This yields a simplified equation for the inverse gluon propagator in Euclidean momentum space,

$$D^{-1\,\mu\nu}(k) = D_0^{-1\,\mu\nu}(k) \quad + \quad \frac{g_0^2 N_C}{32\pi^4 k^2} \int d^4q \, \Gamma_0^{\mu\rho\alpha}(k, q-k, -q) \times D^{\alpha\beta}(q) D_0^{\rho\sigma}(k-q) \Gamma_0^{\beta\sigma\nu}(q, k-q, -k) \quad , \tag{1}$$

where D_0 and Γ_0 are the bare gluon propagator and the bare three-gluon vertex. The use of a second bare vertex in the gluon loop is combined with one of the gluon propagators being bare. This is to account for some of the dressing of the vertex as entailed by its Slavnov-Taylor identity.[1]

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In a manifestly gauge invariant formulation the Dyson–Schwinger equation for the inverse gluon propagator in the covariant gauge would be transverse without further adjustments. This may be violated due to the neglect of ghosts, the violation of Slavnov–Taylor identities and also due to a regularisation that does not preserve the residual local invariance under transformations generated by harmonic gauge functions ($\partial^2\Lambda(x) = 0$). The latter is the case for an $O(4)$ invariant Euclidean cutoff Λ, which we will use to regularise eq. (1). Contracting eq. (1) with the transversal projector,

$$P^{\mu\nu}(k) = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \; . \tag{2}$$

leads to an equation for the gluon renormalisation function $G(k^2)$ in Landau gauge, which after the angular integrations reads[1,2]

$$\begin{aligned} \frac{1}{G(k^2)} = 1 \quad & + \tfrac{g_0^2}{16\pi^2}\tfrac{1}{k^2}\int_0^{k^2} dq^2 \left(\tfrac{7}{8}\tfrac{q^4}{k^4} - \tfrac{25}{4}\tfrac{q^2}{k^2} - \tfrac{9}{2}\right) G(q^2) \\ & + \tfrac{g_0^2}{16\pi^2}\tfrac{1}{k^2}\int_{k^2}^{\Lambda^2} dq^2 \left(\tfrac{7}{8}\tfrac{k^4}{q^4} - \tfrac{25}{4}\tfrac{k^2}{q^2} - \tfrac{9}{2}\right) G(q^2) \; . \end{aligned} \tag{3}$$

However, the above equation contains a quadratically ultraviolet divergent term, which has to be subtracted by a suitable counter term. Generally, quadratic ultraviolet divergences can occur only in the part of the inverse gluon propagator proportional to $\delta^{\mu\nu}$. Therefore, that part cannot be unambiguously determined, it depends on the routing of the momenta. This is due to the various violations of gauge invariance mentioned above. The unambiguous term proportional to $k^\mu k^\nu$ can be obtained by contracting (1) with

$$R^{\mu\nu}(k) = \delta^{\mu\nu} - 4\,\frac{k^\mu k^\nu}{k^2} \quad . \tag{4}$$

In this case, upon angle integration instead of (3) one obtains[3]

$$\begin{aligned} \frac{1}{G(k^2)} = 1 \quad & + \tfrac{g_0^2}{16\pi^2}\tfrac{1}{k^2}\int_0^{k^2} dq^2 \left(\tfrac{7}{2}\tfrac{q^4}{k^4} - \tfrac{17}{2}\tfrac{q^2}{k^2} - \tfrac{9}{8}\right) G(q^2) \\ & + \tfrac{g_0^2}{16\pi^2}\tfrac{1}{k^2}\int_{k^2}^{\Lambda^2} dq^2 \left(\tfrac{7}{8}\tfrac{k^4}{q^4} - 7\tfrac{k^2}{q^2}\right) G(q^2). \end{aligned} \tag{5}$$

The logarithmic ultraviolet divergences in eqs. (3) and (5) can be removed by multiplicative renormalisation. Introducing renormalised gluon propagator and coupling the renormalisation constants Z_3 and Z_g are defined by $D \to Z_3 D$ and $g_0 = Z_g g$. The renormalised Dyson–Schwinger equation for the gluon propagator in Mandelstam approximation then reads,

$$G(k^2) = \left[Z_3 + Z_3^2 Z_g^2 \, \frac{g^2}{16\pi^2} \, I_G(k^2) \right]^{-1} \quad , \tag{6}$$

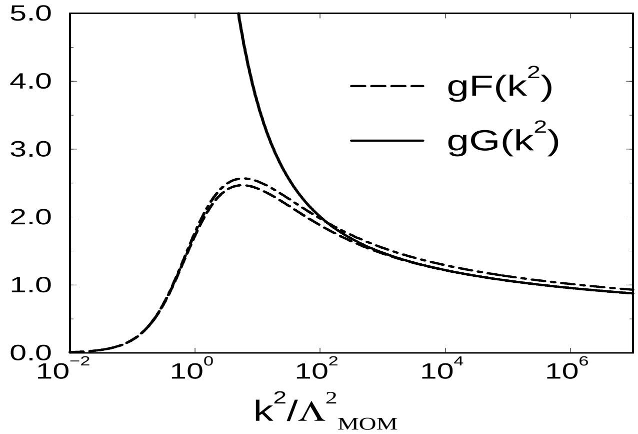

Figure 1: The gluon renormalisation function $gG(k^2) = 8\pi\sigma/k^2 + gF(k^2)$ for eq. 5. The dashed lines show $gF(k^2)$ for eqs. 5 and 3.

with obvious definitions of I_G as a functional of G for the respective cases (*cf.*, eqs. 3 and 5). In Mandelstam approximation one has $Z_g Z_3 = 1$.[4] We adopt a momentum subtraction scheme requiring the gluon self–energy to vanish at the renormalisation point μ: $G(\mu^2) = 1$.

The behaviour of $G(k^2)$ for $k^2 \to 0$ can be summarized as follows:

$$gG(k^2) \,=\, \frac{8\pi\sigma}{k^2} + gF(k^2) \quad , \qquad gF(k^2) = a_{00}\left(k^2/\Lambda^2\right)^\gamma + \dots \quad . \tag{7}$$

Here, σ is the string tension, and the subleading term is determined by[1,2]

$$\gamma = \sqrt{\frac{31}{6}} - 1 \approx 1.273, \quad \text{and} \quad a_{00} \approx 0.29421$$

for Mandelstam's original equation (3). For eq. (5), we obtain,

$$\gamma = \frac{2}{9}\sqrt{229}\cos\left(\frac{1}{3}\arccos\left(-\frac{1099}{229\sqrt{229}}\right)\right) - \frac{13}{9} \approx 1.271\,, \text{ and } \quad a_{00} \approx 0.29446\,.$$

The numerical solutions to eqs. (5) and (3) for the renormalisation group invariant products gG and gF as functions of $k^2/\Lambda^2_{\mathrm{MOM}}$ are shown in fig. 1. As there are no qualitative and only little quantitative differences between those two solutions, we concentrate on the discussion of the solution to eq. (5) in the following.

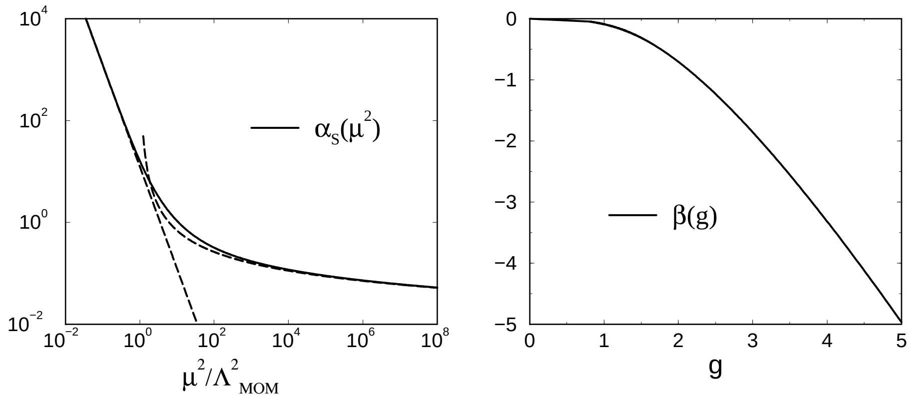

Figure 2: The strong coupling $\alpha_S(\mu^2)$ with its asymptotic limits and the non-perturbative β function of the Mandelstam approximation for a momentum subtraction scheme.[4]

The non-perturbative and renormalisation group invariant result relates the string tension σ to the QCD scale (the only parameter) by $\sigma = 2\Lambda^2$. Fixing σ to its phenomenological value (it may be extracted from heavy quarkonium spectra or the slope of Regge trajectories) gives a value for Λ of about 600 MeV. Alternatively, from $\alpha_S(m_Z^2) \simeq 0.108$ we find $\Lambda \simeq 640$MeV. For a momentum subtraction scheme with $N_f = 0$ this is the correct order of magnitude.

The scaling behaviour of the solution near the ultraviolet fixed point is determined by the coefficients $\beta_0 = 14$, $\beta_1 = 70/3$ and $\gamma_A^0 = 7$ which are reasonably close to the perturbative values for $N_f = 0$, *i.e.*, $\beta_0 = 11$, $\beta_1 = 51$ and $\gamma_A^0 = 13/2$.

The running coupling, which is obtained for arbitrary scales from the renormalisation condition, resembles the two-loop perturbative form asymptotically. It is shown together with the corresponding Callan–Symanzik β function in fig. 2. The running coupling increases without bound in the infrared, no further fixed point exists.

References

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